https://www.linkedin.com/feed/update/urn:li:activity:6548138571155542017
262.3 (NguenDuy Lien) The sequence (a_n) is defined by
$a_0 = 2, a_{n+1} = 4a_n + \sqrt{15a_n^2 - 60}$, for $n \in \mathbb{N}$.
Find the general term a_n. Prove that $\frac{1}{5}(a_{2n} + 8)$ can be represented as the sum of squares of three consecutive integers for $n \geq 1$.

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

First note that $2 \leq a_n < a_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$ ($a_0 = 2$ and $a_{n+1} > 4a_n$).
Also note that $a_1 = 4a_0 + \sqrt{15a_0^2 - 60} = 8 + \sqrt{15 \cdot 2^2 - 60} = 8$.
Since $a_{n+1} = 4a_n + \sqrt{15a_n^2 - 60} \Leftrightarrow (a_{n+1} - 4a_n)^2 = 15a_n^2 - 60 \Leftrightarrow$
$a_{n+1}^2 - 8a_{n+1}a_n + a_n^2 = -60, \forall n \in \mathbb{N} \cup \{0\}$ then
$a_{n+2}^2 - 8a_{n+2}a_{n+1} + a_{n+1}^2 - (a_{n+1}^2 - 8a_{n+1}a_n + a_n^2) = 0 \Leftrightarrow$
$(a_{n+2} - a_n)(a_{n+2} - 8a_{n+1} + a_n) = 0 \Leftrightarrow a_{n+2} - 8a_{n+1} + a_n = 0, n \in \mathbb{N} \cup \{0\}$
and, therefore, $a_n = c_1\left(4 + \sqrt{15}\right)^n + c_2\left(4 - \sqrt{15}\right)^n$.
Initial conditions $a_0 = 2, a_1 = 8$ give us $c_1 = c_2 = 1$.
Thus, $a_n = \left(4 + \sqrt{15}\right)^n + \left(4 - \sqrt{15}\right)^n$, $n \in \mathbb{N} \cup \{0\}$.
We will prove that there is sequence of integer numbers (b_n) such that $\frac{1}{5}(a_{2n} + 8) =$
$(b_n - 1)^2 + b_n^2 + (b_n + 1)^2 = 3b_n^2 + 2 \Leftrightarrow a_{2n} + 8 = 15b_n^2 + 10 \Leftrightarrow a_{2n} - 2 = 15b_n^2$.
Note that $a_{2n} - 2 = \left(4 + \sqrt{15}\right)^{2n} + \left(4 - \sqrt{15}\right)^{2n} - 2 = \left(\left(4 + \sqrt{15}\right)^n - \left(4 - \sqrt{15}\right)^n\right)^2$.
Let $b_n = \dfrac{\left(4 + \sqrt{15}\right)^n - \left(4 - \sqrt{15}\right)^n}{\sqrt{15}}, n \in \mathbb{N} \cup \{0\}$.
Then $b_0 = 0, b_1 = 2,\ b_{n+1} - 8b_n + b_{n-1} = 0, n \in \mathbb{N}$ and, therefore, $a_{2n} - 2 = 15b_n^2$,
$n \in \mathbb{N} \cup \{0\}$ where b_n is obviously integer for any $n \in \mathbb{N} \cup \{0\}$(by Math Induction using $b_0 = 0, b_1 = 2$ as the Base of MI and $b_{n+1} = 8b_n - b_{n-1}, n \in \mathbb{N}$ for the Step of MI).